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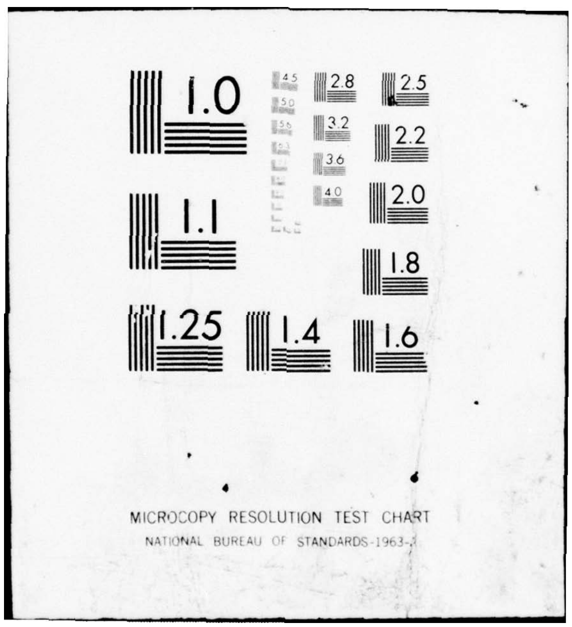
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January 1979

A SUGGESTION AS TO A GENERAL DERIVATION OF THE EQUATIONS OF MOTION OF A DEFORMABLE AIRCRAFT FOR SMALL PERTURBATIONS WHICH WILL BE MOST GENERALLY ACCEPTABLE

by

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REPORT DOCUMENTATION PAGE

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by

(10) D. L. Woodcock

(11) Jan 79

SUMMARY

A general derivation of the equations of motion of a deformable aircraft for small perturbations is offered which it is hoped will be acceptable to all traditions and inclinations. The deformations are represented by an expression which is precisely linear in the generalised coordinates. Lagrange's equation for an arbitrary non-inertial frame is used, along with the principles of momentum, and it is shown how the resulting equations of motion can be particularised to suit various tastes. An Appendix considers to some extent the determination of the phugoid and shows that the traditional structural dynamicist's approach is just as adequate as that of the flight dynamicist.

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1 INTRODUCTION

None of the recent papers from the RAE[1-8], or elsewhere (*eg* Refs 9 and 10), on the formulation of the equations of motion of a deformable aircraft, can be said to have been received enthusiastically or even given general satisfaction. There has been criticism for incomprehensibility and for naïvety; notation and nomenclature have sometimes been off-putting; and there have been particular criticisms of such things as the representation of the perturbations, the choice of frames of reference etc. What is suggested here is a general approach which, it is believed, is simple to follow and can be particularized to suit the whim or fancy of the majority. Thus one can take the principal frame of reference to be body-fixed axes*, mean-body axes*, constant-velocity axes*, or a non-inertial frame satisfying some other (or none at all) conditions.

We will restrict ourselves to small perturbations of a datum motion of constant linear velocity and zero angular velocity, to flight in a uniform atmosphere over a 'flat earth', and to flight of an aircraft of constant mass which is such that no part of it moves relative to any other part during the datum motion. Refinements, such as allowing for the effect of engine rotation, taking other datum motions during which there is no deformation, etc, can be made on the lines developed elsewhere (Refs 6 and 8, for example).

The approach is briefly as follows. We take as a basic frame of reference constant-velocity axes* which are such that they coincide, during the datum motion, with some axes, fixed in the aircraft during the datum motion, such as body-path axes* or body-fixed axes* (perhaps the principal axes of inertia*). The perturbation is then represented by a perturbation with no deformation which moves a set of body-fixed axes from coincidence with the constant-velocity axes to an 'arbitrary' position occupied by axes which we call no-deformation-body-fixed axes, followed by deformations which are such that the translational displacement of any particle relative to the no-deformation-body-fixed axes and referred to those axes is precisely a linear function of a set of generalised coordinates. The equations of motion are then obtained using Lagrange's equation for a non-inertial frame, to obtain the equations for the deformational freedoms; and applying the principles of linear and angular momentum in the directions of and about the no-deformation-body-fixed axes, to obtain the rigid body equations.

Some may say that "we want to treat all the perturbations in the same way, we don't see any point in distinguishing between rigid body perturbations and

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011 * These are defined in the Glossary at the end of the paper.

deformations". The answer is that you do not have to. You merely take the particular case where the no-deformation-body-fixed axes coincide with the constant-velocity axes. That is, put the specific rigid body perturbations to zero and include in the deformational freedoms modes which are exactly (for the translations), or a sufficiently good approximation to (for the rotations), rigid body perturbations.

The advantages of this approach are:

(i) The principal frame of reference - the no-deformation-body fixed axes can be made any non-inertial frame* that one desires, or left, in a sense, arbitrary, by a suitable choice of deformation modes. Thus, for example, modes satisfying certain conditions will make it body-fixed axes, while modes satisfying certain other conditions will make it mean-body axes. Alternatively, as already described, we can make the principal frame of reference an inertial frame - constant-velocity axes.

(ii) The generalised applied forces that appear in Lagrange's equation are obtained by calculating the virtual work done in infinitesimal displacements assuming the frame of reference (the non-inertial frame which is our principal frame of reference) remains stationary**. With our suggested approach the contribution from any applied force to the first order terms (in the generalised coordinates) in the generalised forces is, since the deformation is precisely linear in the generalised coordinates, given merely by the virtual work done by the first order perturbations in the components of the applied force referred to the no-deformation-body-fixed axes. Previous developments (*eg* Ref 2) have been criticised because the required first order terms in the generalised forces involved the unperturbed value of the applied force - this did some significant virtual work since the relevant deformations were not precisely linear in the generalised coordinates. The same criticism could have been made in our present development if we had applied Lagrange's equation for an inertial frame (the constant-velocity axes) without making the specific rigid body rotations all zero.

(iii) The overall applied forces that appear in the other equations of motion - the rigid body equations - are also such that there is no contribution, from the unperturbed value of an applied force, to their linear constituents (coefficients of the generalised coordinates).

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* As already stated it can be made an inertial frame by omission of the specific body freedoms.

** That is that the position of the frame of reference is not affected by the infinitesimal displacements.

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In this paper the development of the equations of motion is made entirely in a dimensional form. There is however little difficulty in writing them in a non-dimensional form, as, for example, was done in Ref 3; though some would question whether anything is gained by so doing rather than working with the dimensional form and a system of units* appropriate to the problem being considered.

2 THE SEMI-RIGID MODEL

Let the constant-velocity frame of reference be identified by constant-velocity axes $O_c x_c y_c z_c$. The location of the no-deformation-body-fixed axes $O_n x_n y_n z_n$ can then be specified by the translations and rotations which, when applied to the constant-velocity axes, would make the two sets of axes coincident. Let these translations and rotations be, in succession,:

(i) Translations $x_1^{(c)}$, $y_1^{(c)}$, $z_1^{(c)}$ in the directions of the respective constant-velocity axes.

(ii) Successive rotations ψ, θ, ϕ about the carried axes O_z, O_y, O_x .

These translations and rotations will be taken as the six specific 'body-freedom' generalised coordinates**. They uniquely specify the position of the no-deformation-body-fixed axes. Following these rigid body perturbations we assume the aircraft is deformed so that the position of a particle, relative to the origin of the no-deformation-body-fixed axes, is given precisely by

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = S \left\{ \begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} - \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \right\} \qquad (2\text{-}1)$$

where R is a modal matrix (3 × n) whose elements are functions of the coordinates (x_f, y_f, z_f) of the unperturbed particle, and S is the axes transformation matrix (see equation (2-4) below).

Writing the linear velocity components of the constant-velocity axes, referred to those axes, as $\{u_f, v_f, w_f\}$, and remembering that their angular

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* Such as the aero-normalised system proposed by Hopkin[14].

** However, if desired, as stated in the Introduction, these steps just described can be omitted and good approximations to the body-freedoms included amongst the deformational freedoms.

velocity is zero, it is easily seen (Ref 1, Appendix A) that the linear and angular velocities of the no-deformation-body-fixed axes are, referred to those axes,

$$\begin{bmatrix} u^{(n)} \\ v^{(n)} \\ w^{(n)} \end{bmatrix} = S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + S \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} \tag{2-2}$$

$$\begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} = Q_\phi \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{2-3}$$

where* S is the axes transformation matrix

$$S = I - A_\phi + \dots \tag{2-4}$$

and

$$Q_\phi = I - J_\phi - \dots \tag{2-5}$$

in which, for any column vector $\{\phi\ \theta\ \psi\}$

$$A_\phi = \begin{bmatrix} 0 & -\psi & \theta \\ \psi & 0 & -\phi \\ -\theta & \phi & 0 \end{bmatrix} \tag{2-6}$$

and

$$J_\phi = \begin{bmatrix} 0 & 0 & \theta \\ 0 & 0 & -\phi \\ 0 & \phi & 0 \end{bmatrix} \tag{2-7}$$

Consequently the linear velocity of a particle, referred to the no-deformation body-fixed axes, is given by (*cf* Ref 6, for example).

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* See Ref 14, Appendix M and Ref 1, Appendix A; and note that S is orthogonal $S^T = S^{-1}$.

$$\begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix} = S\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + S\begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + R\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + A_{p^{(n)}}\left\{\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right\}$$

$$= \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + R\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + (A_{u_f} - A_{x_f} D)\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \dots \tag{2-8}$$

where D is the operator d/dt, and, in addition to (2-3) to (2-5), we have made use of the fact that

$$A_\phi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = - A_x \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \tag{2-8a}$$

for any vectors $\{\phi\ \theta\ \psi\}$, $\{x\ y\ z\}$.

2.1 Various particularisations

It has been pointed out in Ref 6 that if the modal matrix R satisfies the conditions

$$\sum \delta m R = \sum \delta m A_{x_f} R = 0 \tag{2-9}$$

where δm is the mass of an element and the summation is for all the elements of the aircraft, then the no-deformation-body-fixed axes are identical with the mean-body axes (satisfying the condition that the kinetic energy relative to the axes is a minimum). This condition (2-9) means that the deformation modes are orthogonal to the specific body freedoms with respect to the mass distribution.

If in addition to (2-9) we impose the condition

$$\sum \delta m \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = 0 \tag{2-10}$$

which makes the origin of the no-deformation-body-fixed axes at the aircraft CG when there is no deformation, then from (2-1), the origin of these axes (alias the mean-body axes) stays at the CG whatever the deformation. Equation (2-10) means that the specific body freedoms are mutually orthogonal with respect to the mass distribution.

Mentioned in the same paper[6] is the case when at the particle $x_f = y_f = z_f = 0$.

$$\left.\begin{array}{ll} R = 0 & \\ \dfrac{\delta R}{\delta x_f} & \text{has null second and third rows} \\ \dfrac{\delta R}{\delta y_f} & \text{has null third and first rows} \\ \dfrac{\delta R}{\delta z_f} & \text{has null first and second rows.} \end{array}\right\} \qquad (2\text{-}11)$$

The no-deformation-body-fixed axes are then identical with body-fixed axes fixed in position and orientation, in the aircraft, at their origin.

Another case of interest occurs when we make the specific body freedom coordinates $x_1^{(c)}$, ϕ etc) all zero and put

$$R = \begin{bmatrix} \tilde{R} & I & -A_{x_f} \end{bmatrix} \qquad (2\text{-}12)$$

The last three of these modes are then good approximations to the rigid body rotations, and the previous three are rigid body translations. Moreover the no-deformation-body-fixed axes are then identical with the constant-velocity axes. The approach then becomes that favoured by many flutter analysts (*cf* Ref 7). With this representation the coordinates of a particle relative to the comstant-velocity axes is

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \tilde{R} \begin{bmatrix} q_1 \\ \vdots \\ q_{n-6} \end{bmatrix} + \begin{bmatrix} q_{n-5} \\ q_{n-4} \\ q_{n-3} \end{bmatrix} - A_{x_f} \begin{bmatrix} q_{n-2} \\ q_{n-1} \\ q_n \end{bmatrix} \qquad (2\text{-}13)$$

which compares with our basic representation

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + S_\phi^T \left\{ \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\}$$

$$= \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} - A_{x_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \dots \quad . \qquad (2\text{-}14)$$

Thus the two representations agree for small perturbations if, in (2-13), we replace n by $n + 6$, and then put

$$\left. \begin{aligned} \tilde{R} &= R \\ \begin{bmatrix} q_{n+1} \\ q_{n+2} \\ q_{n+3} \end{bmatrix} &= \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \\ \begin{bmatrix} q_{n+4} \\ q_{n+5} \\ q_{n+6} \end{bmatrix} &= \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \end{aligned} \right\} \quad . \qquad (2\text{-}15)$$

On occasion it may be desirable or convenient to specify the deformation of the aircraft in terms of translations and rotations of strips of the aircraft and of rotations of 'flap' parts of the strips relative to the main parts of their own strips. It will be seen by comparison with the analysis of Ref 5, or otherwise, that a good approximation to such a representation is achieved by taking

$$R = K_i - S_{\phi_{if}}^T A_{x_{sif}} Q_{\phi_{if}} F_i + S_{\phi_{if}}^T \begin{bmatrix} z_{sif} \\ 0 \\ x_{sif} - x_{hsi} \end{bmatrix} f_i^T \qquad (2\text{-}16)$$

at the ith strip.

In this expression the ϕ_{if} etc express the orientation of some strip-attached axes relative to the no-deformation-body-fixed axes in the datum state*; the x_{sif} etc are the datum coordinates of a particle in the strip-attached frame of reference; $x_{sif} = x_{hsi}$ $(y_{sif} = z_{sif} = 0)$ is the hinge position; and K_i, F_i, f_i are respectively translational rotational (approximate) and flap rotational (approximate) modal matrices. For our present purpose we will simplify the above expression by assuming that in the datum state all the strip-attached axes have the same orientation as the no-deformation-body-fixed axes in the datum state. Thus equation (2-16) becomes then

$$R = K_i - A_{x_{sif}} F_i + \begin{bmatrix} z_{sif} \\ 0 \\ x_{sif} - x_{hsi} \end{bmatrix} f_i^T \qquad (2\text{-}17)$$

at the ith strip. This is an extreme simplification. A more representative model would be to take

$$\begin{bmatrix} \phi_{if} \\ \theta_{if} \\ \psi_{if} \end{bmatrix} = \text{either} \begin{bmatrix} 0 \\ \theta_{if} \\ 0 \end{bmatrix} \text{or} \begin{bmatrix} \pi/2 \\ 0 \\ 0 \end{bmatrix} \qquad (2\text{-}18)$$

where the second option would be used for strips such as those on a fin. With a modal matrix of the form (2-17), since in this case

$$\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + \begin{bmatrix} x_{sif} \\ y_{sif} \\ z_{sif} \end{bmatrix} \qquad (2\text{-}19)$$

* The strip-attached axes are attached to the strip at the strip reference point, which is in the main part, not the flap part, of the strip. During the datum motion they are fixed in the strip. During perturbed motion, according to the above modal matrix R (equation (2-16)), the deformation of the strip will only differ, from a deformation which is purely a rotation of the flap part of the strip, by terms which are of second order in the generalised coordinates. Thus, when the perturbations are small, we can say that the strip-attached axes are almost fixed in the main part of the strip. We are assuming, by definition, that the flap angle is zero in the datum state. Ref 5 is more general being written in terms of datum values of the flap angle. 011

where (x_{if}, y_{if}, z_{if}) is the strip reference point in the datum state, we can include good approximations to the rigid body modes, as in (2-12), by taking

$$K_i = \begin{bmatrix} \tilde{K}_i & I & -A_{x_{if}} \end{bmatrix} \tag{2-20}$$

$$F_i = \begin{bmatrix} \tilde{F}_i & 0 & I \end{bmatrix} \tag{2-21}$$

and

$$f_i^T = \begin{bmatrix} \tilde{f}_i^T & 0 & 0 \end{bmatrix} . \tag{2-22}$$

In the more general representation (2-16), since then

$$\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} x_{sif} \\ y_{sif} \\ z_{sif} \end{bmatrix} \tag{2-23}$$

one would have, instead of (2-21), to take

$$F_i = \begin{bmatrix} \tilde{F}_i & 0 & Q^{-1}_{\phi_{if}} S_{\phi_{if}} \end{bmatrix} . \tag{2-24}$$

3 THE APPLIED FORCES

Let a typical local applied force, referred to the no-deformation-body-fixed axes, be (the bar suprascript denotes typical)

$$\begin{bmatrix} \bar{e}^{(n)} \\ \bar{f}^{(n)} \\ \bar{g}^{(n)} \end{bmatrix} = \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_n \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \tag{3-1}$$

where the coefficients $\bar{e}_j$, $\bar{e}_x$, $\bar{e}_\phi$ etc may be differential operators. In the following two sub-sections we give the contributions, from a set of such forces,

to the overall and generalised applied forces. Then in sections 4 to 6 we obtain these contributions when applied forces are, respectively, aerodynamic gravitational and structural. Since this paper is intentionally short, propulsive forces have not been considered separately - they were considered in Refs 1 to 6 - it being left to the reader to do that, or, alternatively, to include them with the aerodynamic forces.

For the particular case (equation (2-12)) mentioned in section 2.1, when n is increased to $n + 6$ and $\tilde{R}$ put equal to R, equation (3-1) becomes

$$\begin{bmatrix} \bar{e}^{(n)} \\ \bar{f}^{(n)} \\ \bar{g}^{(n)} \end{bmatrix} = \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \left[\begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_n \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} - A_{\bar{e}_f} \right] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix}$$

$$+ \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \cdots \quad . \qquad (3\text{-}2)$$

The difference between the coefficients of $\{q_{n+4} \ q_{n+5} \ q_{n+6}\}$ and $\{\phi \ \theta \ \psi\}$ is due to the fact that the no-deformation-body-fixed axes are differently oriented, when the first of these is zero and not the second, compared with the case when the second is zero and not the first*. Of course when one takes the above form one will normally make all the specific body freedom coordinates ($x_1^{(c)}$, θ etc) zero.

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* In this latter case the no-deformation-body-fixed axes coincide with the constant-velocity axes, while in general they do not. The coefficients of q_{n+1} to q_{n+6} in (3-2) were therefore found from the condition

$$S\left\{ \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{e}_{n+1} & \cdots & \bar{e}_{n+6} \\ \bar{f}_{n+1} & \cdots\cdots & \\ \bar{g}_{n+1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi \\ \theta \\ \psi \end{bmatrix} \right\} \approx \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

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......(3-2a)

3.1 The generalised applied forces

By considering the virtual work done by a set of forces (3-1) in infinitesimal displacements, assuming the no-deformation-body-fixed axes to be stationary* (*cf* equation (2-1)), their contributions to the generalised applied forces are easily seen to be

$$\begin{bmatrix}\bar{Q}_1\\ \vdots \\ \bar{Q}_n\end{bmatrix} = \sum R^T \begin{bmatrix}\bar{e}^{(n)}\\ \bar{f}^{(n)}\\ \bar{g}^{(n)}\end{bmatrix} = \sum R^T \begin{bmatrix}\bar{e}_f\\ \bar{f}_f\\ \bar{g}_f\end{bmatrix} + \sum R^T \begin{bmatrix}\bar{e}_1 & \cdots & \bar{e}_n\\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix}\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix} + \sum R^T \begin{bmatrix}\bar{e}_x & \bar{e}_y & \bar{e}_z\\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix}\begin{bmatrix}x_1^{(c)}\\ y_1^{(c)}\\ z_1^{(c)}\end{bmatrix}$$

$$+ \sum R^T \begin{bmatrix}\bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi\\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix} + \cdots \quad . \tag{3-3}$$

3.2 The overall applied forces

The contributions from the typical set of applied forces (3-1) to the translational force on the aircraft, and moment about the origin of the no-deformation-body-fixed axes, referred to those axes, are respectively

$$\begin{bmatrix}\bar{X}^{(n)}\\ \bar{Y}^{(n)}\\ \bar{Z}^{(n)}\end{bmatrix} = \sum \begin{bmatrix}\bar{e}^{(n)}\\ \bar{f}^{(n)}\\ \bar{g}^{(n)}\end{bmatrix}$$

$$= \sum \begin{bmatrix}\bar{e}_f\\ \bar{f}_f\\ \bar{g}_f\end{bmatrix} + \sum \begin{bmatrix}\bar{e}_1 & \cdots & \bar{e}_n\\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix}\begin{bmatrix}q_1\\ \vdots \\ q_n\end{bmatrix}$$

$$+ \sum \begin{bmatrix}\bar{e}_x & \bar{e}_y & \bar{e}_z\\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix}\begin{bmatrix}x_1^{(c)}\\ y_1^{(c)}\\ z_1^{(c)}\end{bmatrix} + \sum \begin{bmatrix}\bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi\\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix}\begin{bmatrix}\phi\\ \theta\\ \psi\end{bmatrix} + \cdots \tag{3-4}$$

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* That is that the position of the frame of reference is not affected by the infinitesimal displacements.

and

$$\begin{bmatrix} \bar{L}_n^{(n)} \\ \bar{M}_n^{(n)} \\ \bar{N}_n^{(n)} \end{bmatrix} = \sum A_{x_n}^{(n)} \begin{bmatrix} \bar{e}^{(n)} \\ \bar{f}^{(n)} \\ \bar{g}^{(n)} \end{bmatrix}$$

$$= \sum A_{x_f} \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \sum \left\{ A_{x_f} \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_n \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} - A_{\bar{e}_f} R \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \sum A_{x_f} \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + \sum A_{x_f} \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \cdots \quad . \tag{3-5}$$

In the particular case, (equation (2-12)), mentioned in section 2.1, when n is increased to $n + 6$ and $\tilde{R}$ put equal to R, the above two expressions become, using (3-2) instead of (3-1)

$$\begin{bmatrix} \bar{X}^{(n)} \\ \bar{Y}^{(n)} \\ \bar{Z}^{(n)} \end{bmatrix} = \sum \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \left[\sum \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_n \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \quad \sum \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \quad \sum \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} - A_{\bar{X}_f} \right] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix}$$

$$+ \sum \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \cdots\cdots & \\ \bar{g}_x & \cdots\cdots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + \sum \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \cdots\cdots & \\ \bar{g}_\phi & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \cdots \tag{3-6}$$

$$\begin{bmatrix} \bar{L}_n^{(n)} \\ \bar{M}_n^{(n)} \\ \bar{N}_n^{(n)} \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \sum \left\{ A_{x_f} \begin{bmatrix} \bar{e}_1 & \dots & \bar{e}_n \\ \bar{f}_1 & \dots\dots & \\ \bar{g}_1 & \dots\dots & \end{bmatrix} - A_{\bar{e}_f} R \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \left\{ \sum A_{x_f} \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \dots\dots & \\ \bar{g}_x & \dots\dots & \end{bmatrix} - A_{\bar{X}_f} \right\} \begin{bmatrix} q_{n+1} \\ q_{n+2} \\ q_{n+3} \end{bmatrix}$$

$$+ \left\{ \sum A_{x_f} \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \dots\dots & \\ \bar{g}_\phi & \dots\dots & \end{bmatrix} - A_{\bar{L}_f} \right\} \begin{bmatrix} q_{n+4} \\ q_{n+5} \\ q_{n+6} \end{bmatrix}$$

$$+ \sum A_{x_f} \begin{bmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ \bar{f}_x & \dots\dots & \\ \bar{g}_x & \dots\dots & \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + \sum A_{x_f} \begin{bmatrix} \bar{e}_\phi & \bar{e}_\theta & \bar{e}_\psi \\ \bar{f}_\phi & \dots\dots & \\ \bar{g}_\phi & \dots\dots & \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \dots \quad . \tag{3-7}$$

4 THE AERODYNAMIC FORCES

The boundary condition to be satisfied by the air motion at the surface of the body involves the velocity of the surface normal to itself. The velocity vector of a particle is given by equation (2-8) in terms of its components referred to the no-deformation-body-fixed axes. The slope vector of the body surface relative to the same axes will be merely a function of the generalised coordinates q_1 to q_n (*cf* equation (2-1)). The local aerodynamic force vector will therefore be a function of the three quantities

$$S\left\{\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix}\right\}, \qquad Q_\phi \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}, \qquad \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{4-1}$$

and their derivatives with respect to time. As in previous papers[1-6] we will take a model involving no hereditary constituent. We therefore write the local aerodynamic force vector as*

$$\begin{bmatrix} e_a^{(n)} \\ f_a^{(n)} \\ g_a^{(n)} \end{bmatrix} = \begin{bmatrix} e_{af} \\ f_{af} \\ g_{af} \end{bmatrix} + \begin{bmatrix} e_{a1} & \cdots & e_{an} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} \left\{ \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$+ \ldots \qquad (4\text{-}2)$$

where the coefficients e_{aj}, $e_{\dot{x}}$, $e_{\dot{\phi}}$ etc may be differential operators. Putting this expression into equation (3-3) we find that the aerodynamic contributions to the generalised applied forces are

$$\begin{bmatrix} Q_{a1} \\ \vdots \\ Q_{an} \end{bmatrix} = \sum R^T \begin{bmatrix} e_{af} \\ f_{af} \\ g_{af} \end{bmatrix} + \left(\sum R^T \begin{bmatrix} e_{a1} & \cdots & e_{an} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left(\sum R^T \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} \right) \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix}$$

$$+ \left(\sum R \left\{ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \ldots \quad . \qquad (4\text{-}3)$$

Furthermore the translational force on the aircraft, and the moment about the origin of the no-deformation-body-fixed axes, resulting from the aerodynamic load distribution (4-2), are, respectively, (*cf* equation (3-4) and (3-5)).

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* The subscript a , indicating aerodynamic, has only been introduced where it might possibly serve a useful purpose. It was always omitted in Refs 1 to 6.

$$\begin{bmatrix} X_a^{(n)} \\ Y_a^{(n)} \\ Z_a^{(n)} \end{bmatrix} = \sum \begin{bmatrix} e_{af} \\ f_{af} \\ g_{af} \end{bmatrix} + \left(\sum \begin{bmatrix} e_{a1} & \cdots & e_{an} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left(\sum \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} \right) \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix}$$

$$+ \left(\sum \left\{ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \cdots$$

$$= (\text{say}) \begin{bmatrix} X_{af} \\ Y_{af} \\ Z_{af} \end{bmatrix} + \begin{bmatrix} X_{a1} & \cdots & X_{an} \\ Y_{a1} & \cdots\cdots & \\ Z_{a1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \left\{ \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & Z_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} X_{\dot{\phi}} & X_{\dot{\theta}} & X_{\dot{\psi}} \\ Y_{\dot{\phi}} & \cdots\cdots & \\ Z_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \cdots \qquad (4\text{-}4)$$

and

$$\begin{bmatrix} L_{an}^{(n)} \\ M_{an}^{(n)} \\ N_{an}^{(n)} \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} e_{af} \\ f_{af} \\ g_{af} \end{bmatrix} + \sum \left(A_{x_f} \begin{bmatrix} e_{a1} & \cdots & e_{an} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} - A_{e_{af}} R \right) \begin{bmatrix} q_1 \\ \vdots \\ q_4 \end{bmatrix} + \left(\sum A_{x_f} \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} \right) \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix}$$

$$+ \left(\sum A_{x_f} \left\{ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \cdots$$

$$= (\text{say}) \begin{bmatrix} L_{af} \\ M_{af} \\ N_{af} \end{bmatrix} + \begin{bmatrix} L_{a1} & \cdots & L_{an} \\ M_{a1} & \cdots\cdots & \\ N_{a1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \left\{ \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\phi}} & L_{\dot{\theta}} & L_{\dot{\psi}} \\ M_{\dot{\phi}} & \cdots\cdots & \\ N_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \cdots \quad . \qquad (4\text{-}5)$$

The coefficients X_{a_1}, $X_{\dot{x}}$ etc in the above two equations, (4-4) and (4-5), may be differential operators.

For the particular case (equation (2-12)) mentioned in section 2.1, when n is increased to $n + 6$, $\tilde{R}$ is put equal to R , and the specific body freedom coordinates made zero, then the expression for the local aerodynamic force vector has the form (*cf* equations (4-2) and (3-2))

$$\begin{bmatrix} e_a^{(n)} \\ f_a^{(n)} \\ g_a^{(n)} \end{bmatrix} = \begin{bmatrix} e_{af} \\ f_{af} \\ g_{af} \end{bmatrix} + \begin{bmatrix} e_{a1} & \cdots & e_{a,n+6} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} + \cdots \qquad (4\text{-}6)$$

where

$$\begin{bmatrix} e_{a,n+1} & \cdots & e_{a,n+3} \\ f_{a,n+1} & \cdots\cdots & \\ g_{a,n+1} & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} D \qquad (4\text{-}7)$$

and

$$\begin{bmatrix} e_{a,n+4} & \cdots & e_{a,n+6} \\ f_{a,n+4} & \cdots\cdots & \\ g_{a,n+4} & \cdots\cdots & \end{bmatrix} = \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D - A_{e_{af}} \; . \qquad (4\text{-}8)$$

The generalised forces are then given by the obvious modification of (4-3) - $x_1^{(c)}$, ϕ etc zero, the use of (4-6), and the replacement of R by $[R \quad I \quad -A_{x_f}]$.

4.1 The use of strip aerodynamics

If the modes have been expressed in a strip form, such as (2-17), then the user may find it more convenient or more understandable to have the aerodynamic forces expressed in terms of the overall force, moment etc on a strip. Assuming two-dimensional aerodynamics for each strip, and no inter-strip interference*, we can write[5] the overall forces on a strip, when in the unperturbed state the strip-attached axes have the same orientation as the constant-velocity axes, as

$$\begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} = \begin{bmatrix} X_{if} \\ 0 \\ Z_{if} \end{bmatrix} + \begin{bmatrix} X_{i\dot{x}} & X_{i\dot{z}} & X_{i\dot{\theta}} & X_{i\delta} \\ 0 & 0 & 0 & 0 \\ Z_{i\dot{x}} & Z_{i\dot{z}} & Z_{i\dot{\theta}} & Z_{i\delta} \end{bmatrix} \begin{bmatrix} u_i^{(s)} - u_f \\ w_i^{(s)} - w_f \\ q_i^{(s)} \\ \delta_i \end{bmatrix} + \cdots \qquad (4\text{-}9)$$

where the superscript (s) indcates the resolution along the strip-attached axes; $u_i^{(s)}$, $w_i^{(s)}$ are components of the velocity of the strip reference point; and $q_i^{(s)}$

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* Not even between two strips where one lies aft of the other. For the purpose of this paper things are being kept simple.

is a component of the angular velocity of the strip-attached axes; and the coefficients $X_{i\dot{x}}$, $Z_{i\dot{x}}$ etc may be differential operators. Thus, from (2-8) to (2.17)

$$\begin{bmatrix} u_i^{(s)} \\ v_i^{(s)} \\ w_i^{(s)} \end{bmatrix} = \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + \left(A_{u_f} F_i + K_i D\right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \left(A_{u_f} - A_{x_{if}} D\right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \qquad (4\text{-}10)$$

since the axes transformation matrix (no-deformation-body-fixed to strip attached) is

$$I - A \left(F_i \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right) \ldots \quad . \qquad (4\text{-}11)$$

Also

$$\begin{bmatrix} p_i^{(s)} \\ q_i^{(s)} \\ r_i^{(s)} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + F_i \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \ldots \qquad (4\text{-}12)$$

$$\delta_i = f_i^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad . \qquad (4\text{-}13)$$

The deformation of the strip, other than the flap rotation, will only affect higher order terms in (4-9), since, as we have pointed out, the assumed behaviour (equations (2-1) and (2-17)) only differs from that of a rigid aerofoil and a rigid flap by a term which is of second order of smallness. Thus, from (4-9), using (4-11), the overall forces on a strip resolved along the no-deformation-body-fixed axes are

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$$\begin{bmatrix} X_i^{(n)} \\ Y_i^{(n)} \\ Z_i^{(n)} \end{bmatrix} = \begin{bmatrix} X_{if} \\ 0 \\ Z_{if} \end{bmatrix} + \left\{ \begin{bmatrix} X_{i\dot{x}} & 0 & Z_{i\dot{z}} \\ 0 & 0 & 0 \\ Z_{i\dot{x}} & 0 & Z_{i\dot{z}} \end{bmatrix} \left(A_{u_f} F_i + K_i D\right) + \begin{bmatrix} 0 & X_{i\dot{\theta}} & 0 \\ 0 & 0 & 0 \\ 0 & Z_{i\dot{\theta}} & 0 \end{bmatrix} F_i D + \begin{bmatrix} X_{i\delta} \\ 0 \\ Z_{i\delta} \end{bmatrix} f^T - A_{x_{if}} F_i \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \begin{bmatrix} X_{i\dot{x}} & 0 & X_{i\dot{z}} \\ 0 & 0 & 0 \\ Z_{i\dot{x}} & 0 & Z_{i\dot{z}} \end{bmatrix} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \left\{ \begin{bmatrix} X_{i\dot{x}} & 0 & X_{i\dot{z}} \\ 0 & 0 & 0 \\ Z_{i\dot{x}} & 0 & Z_{i\dot{z}} \end{bmatrix} \left(A_{u_f} - A_{x_{if}} D\right) + \begin{bmatrix} 0 & X_{i\dot{\theta}} & 0 \\ 0 & 0 & 0 \\ 0 & Z_{i\dot{\theta}} & 0 \end{bmatrix} D \right\} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \qquad (4\text{-}14)$$

Similarly the overall moment on the strip, about the strip reference point, and the hinge moment, referred to the no-deformation-body-fixed axes have the forms

$$\begin{bmatrix} L_{is}^{(n)} \\ M_{is}^{(n)} \\ N_{is}^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ M_{if} \\ 0 \end{bmatrix} + \left\{ \begin{bmatrix} 0 & 0 & 0 \\ M_{i\dot{x}} & 0 & M_{i\dot{z}} \\ 0 & 0 & 0 \end{bmatrix} \left(A_{u_f} F_i + K_i D \right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{i\dot{\theta}} & 0 \\ 0 & 0 & 0 \end{bmatrix} F_i D + \begin{bmatrix} 0 \\ M_{i\dot{\delta}} \\ 0 \end{bmatrix} f^T - A \begin{bmatrix} 0 \\ M_{if} \\ 0 \end{bmatrix} F_i \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ M_{i\dot{x}} & 0 & M_{i\dot{z}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + \left\{ \begin{bmatrix} 0 & 0 & 0 \\ M_{i\dot{x}} & 0 & M_{i\dot{z}} \\ 0 & 0 & 0 \end{bmatrix} \left(A_{u_f} - A_{x_{if}} D \right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{i\dot{\theta}} & 0 \\ 0 & 0 & 0 \end{bmatrix} D \right\} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$+ \cdots \qquad (4\text{-}15)$$

and, for the hinge moment,

$$\begin{bmatrix} L_{i''h}^{(n)} \\ M_{i''h}^{(n)} \\ N_{i''h}^{(n)} \end{bmatrix} = \text{a similar expression with each } M \text{ replaced by a } B \qquad (4\text{-}16)$$

We can compare these expressions, (4-14), (4-15) and (4-16), with the corresponding expression obtained from (4-2), viz

$$\begin{bmatrix} X_i^{(n)} \\ Y_i^{(n)} \\ Z_i^{(n)} \end{bmatrix} = \sum_{\substack{\text{for the} \\ \text{ith strip}}} \begin{bmatrix} e_a^{(n)} \\ f_a^{(n)} \\ g_a^{(n)} \end{bmatrix} \qquad (4\text{-}17)$$

$$\begin{bmatrix} L_{is}^{(n)} \\ M_{is}^{(n)} \\ N_{is}^{(n)} \end{bmatrix} = \sum_{\substack{\text{for the} \\ \text{ith strip}}} A_{x_{si}^{(n)}} \begin{bmatrix} e_a^{(n)} \\ f_a^{(n)} \\ g_a^{(n)} \end{bmatrix} \qquad (4\text{-}18)$$

$$\begin{bmatrix} L_{i''h}^{(n)} \\ M_{i''h}^{(n)} \\ N_{i''h}^{(n)} \end{bmatrix} = \sum_{\substack{\text{for the} \\ \text{ith strip}}}'' A_{x_{hi}^{(n)}} \begin{bmatrix} e_a^{(n)} \\ f_a^{(n)} \\ g_a^{(n)} \end{bmatrix} \qquad (4\text{-}19)$$

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where (*cf* equations (2-1) and (2-17).

$$\begin{bmatrix} x_{si}^{(n)} \\ y_{si}^{(n)} \\ z_{si}^{(n)} \end{bmatrix} = \begin{bmatrix} x_{sif} \\ y_{sif} \\ z_{sif} \end{bmatrix} + \left\{ - A_{x_{sif}} F_i + \begin{bmatrix} z_{sif} \\ 0 \\ x_{sif} - x_{hsi} \end{bmatrix} f_i^T \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{4-20}$$

$$\begin{bmatrix} x_{hi}^{(n)} \\ y_{hi}^{(n)} \\ z_{hi}^{(n)} \end{bmatrix} = \begin{bmatrix} x_{sif} - x_{hsi} \\ y_{sif} \\ z_{sif} \end{bmatrix} + \left\{ - A_{(x_{sif}-x_{hsi})} F_i + \begin{bmatrix} z_{sif} \\ 0 \\ x_{sif} - x_{hsi} \end{bmatrix} f_i^T \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{4-21}$$

and the " superscript indicates a summation over the flap part of the strip. Carrying out the comparisons enables us to express various stripwise summations, of the coefficients that appear in the expression (equation (4-2)) for the local aerodynamic force vector, in terms of the strip aerodynamic coefficients X_{ix} etc. These summations appear in the expressions for the overall and generalised aerodynamic forces (equations (4-3), (4-4) and (4-5)), bearing in mind that they relate to the case when the modal matrix R has the form (2-17). We will not detail the relationships here, nor will we, in the present paper, give any further consideration to the use of strip theory aerodynamics.

5 THE GRAVITATIONAL FORCES

On each particle of the aircraft there will be, according to the standard assumption, a gravitational force δmg acting in the z_0 direction (vertically downwards) of the normal earth-fixed axes*. If the orientation transformation matrix which takes us from these axes to the constant-velocity axes is S_{Φ_f},

where $$S_{\Phi_f} = I - A_{\Phi_f} + \ldots \tag{5-1}$$

is similar to S ; then the matrix which performs the orientation transformation from the normal earth-fixed axes to the no-deformation-body-fixed axes is SS_{Φ_f}. Thus the local gravitational force vector referred to the no-deformation-body-fixed axes is

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* See Glossary of nomenclature.

$$\begin{bmatrix} e_g^{(n)} \\ f_g^{(n)} \\ g_g^{(n)} \end{bmatrix} = SS_{\Phi_f} \begin{bmatrix} 0 \\ 0 \\ \delta mg \end{bmatrix} = \delta mg \left\{ \ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} \tag{5-2}$$

where ℓ_{Φ_f} is the third column of S_{Φ_f} .

Consequently, from the formulae of section 3, the gravitational contributions to the generalised and overall applied forces are, respectively,

$$-\begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} = g\left(\sum \delta mR^T\right) \left\{ \ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} \tag{5-3}$$

and

$$\begin{bmatrix} X_g^{(n)} \\ Y_g^{(n)} \\ Z_g^{(n)} \end{bmatrix} = mg \left\{ \ell_{\Phi_f} + A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} \tag{5-4}$$

$$\begin{bmatrix} L_{gn}^{(n)} \\ M_{gn}^{(n)} \\ N_{gn}^{(n)} \end{bmatrix} = g \left\{ \left(\sum \delta mA_{x_f}\right) \ell_{\Phi_f} - A_{\ell_{\Phi_f}} \left(\sum \delta mR\right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right.$$

$$\left. + \left(\sum \delta mA_{x_f}\right) A_{\ell_{\Phi_f}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \ldots \right\} \; . \tag{5-5}$$

For the particular case, mentioned in section 2.1 (equation 2-12)), when n is increased to $n + 6$, $\tilde{R}$ put equal to R , and the specific body freedom coordinates made zero, then the expression for the local gravitational force vector has the form (*cf* equations (5-1) and (3-2))

$$\begin{bmatrix} e_g^{(n)} \\ f_g^{(n)} \\ g_g^{(n)} \end{bmatrix} = \delta m g \ell_{\Phi_f} \quad . \tag{5-6}$$

The generalised forces then are

$$-\begin{bmatrix} G_1 \\ \vdots \\ G_n \end{bmatrix} = g\left(\sum \delta m \begin{bmatrix} R^T \\ I \\ A_{x_f} \end{bmatrix}\right)\ell_{\Phi_f} \quad . \tag{5-7}$$

6 THE STRUCTURAL FORCES

Resulting from such things as the stresses in the material, friction between different components of the structure etc, there will be a force on any particle of the aircraft which can be called a structural force. The meaning of this term is considered more fully in Appendix A of Ref 2. This structural force vector will depend only on the deformation of the aircraft and so we can write its components $\left(e_s^{(n)}, f_s^{(n)}, g_s^{(n)}\right)$, referred to the no-deformation-body-fixed axes, in the form

$$\begin{bmatrix} e_s^{(n)} \\ f_s^{(n)} \\ g_s^{(n)} \end{bmatrix} = \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix}\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \ldots \tag{6-1}$$

where the coefficients, e_{si} etc, may be differential operators. This approximates to any contribution not resulting from a stress by a contribution of the above form.

The structure cannot exert any overall force or moment on itself and so it is easily seen (*cf* Appendix A of Ref 2) that the following conditions must be satisfied:

$$\sum \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = 0 \tag{6-2}$$

$$\sum A_{x_f} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} = 0 \tag{6-3}$$

$$\sum \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} = 0 \tag{6-4}$$

$$\sum A_{x_f} \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} = \sum A_{e_{sf}} R \ . \tag{6-5}$$

Consequently from the formulae of section 3 there is no structural contribution to the overall force and moment, while the contribution to the generalised forces is

$$-\begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix} = \sum R^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \cdots \tag{6-6}$$

For the particular case, mentioned in section 2.1 (equation (2-12)), when n is increased to $n + 6$, $\widetilde{R}$ put equal to R, and the specific body freedom coordinates made zero, then the expression for the local stiffness force vector becomes (*cf* equations (6-1) and (3-2))

$$\begin{bmatrix} e_s^{(n)} \\ f_s^{(n)} \\ g_s^{(n)} \end{bmatrix} = \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \begin{bmatrix} e_{s1} & \cdots & e_{s,n+6} \\ f_{s1} & \cdots\cdots\cdots & \\ g_{s1} & \cdots\cdots\cdots & \end{bmatrix} + \cdots \tag{6-7}$$

where

$$\begin{bmatrix} e_{s,n+1} & \cdots & e_{s,n+3} \\ f_{s,n+1} & \cdots\cdots & \\ g_{s,n+1} & \cdots\cdots & \end{bmatrix} = 0 \qquad (6\text{-}8)$$

and

$$\begin{bmatrix} e_{s,n+4} & \cdots & e_{s,n+6} \\ f_{s,n+4} & \cdots\cdots & \\ g_{s,n+4} & \cdots\cdots & \end{bmatrix} = -A_{e_{sf}} \qquad (6\text{-}9)$$

The generalised forces (structural contribution) then are

$$-\begin{bmatrix} E_1 \\ \vdots \\ E_n \end{bmatrix} = \left[\begin{matrix} \sum R^T \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \\ 0 \\ 0 \end{matrix} \right] + \left[\begin{matrix} \sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} & 0 & -\sum R^T A_{e_{sf}} \\ 0 & 0 & 0 \\ \sum A_{e_{sf}} R & 0 & -\sum A_{x_f} A_{e_{sf}} \end{matrix} \right] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} + \cdots$$

...... (6-10)

where use has been made of equations (6-2) to (6-5). Many readers will be amazed to see non-zero terms outside the (11) submatrix in this equation and in particular to see the term $\{\sum A_{x_f} A_{e_{sf}}\}$ in the bottom right hand corner. "How can a structure have an elastic stiffness in degrees of freedom which to all intents and purposes are rigid body rotations?" one asks. Consider the case when the only freedoms are ϕ, θ, ψ. The local structural force vector is then

$$\begin{bmatrix} e_s^{(n)} \\ f_s^{(n)} \\ g_s^{(n)} \end{bmatrix} = \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \qquad (6\text{-}11)$$

referred to the no-deformation-body-fixed axes and

$$\begin{bmatrix} e_s^{(c)} \\ f_s^{(c)} \\ g_s^{(c)} \end{bmatrix} = \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} - A_{e_{sf}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \dots \tag{6-12}$$

referred to the constant-velocity axes. The coordinates of a typical particle referred to the constant-velocity axes are similarly

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ x_c^{(c)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} - A_{x_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + B_{\phi\theta}^T \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + \dots \tag{6-13}$$

where $B_{\phi\theta}$ is the second order term in the expansion of S (*cf* equation (2-4)). Thus the moment of the force distribution (6-12) about the origin of the constant-velocity axes is

$$\begin{aligned} \sum A_{x_c^{(c)}} \begin{bmatrix} e_s^{(c)} \\ f_s^{(c)} \\ g_s^{(c)} \end{bmatrix} &= \sum A_{x_f} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \sum \left(A_{e_{sf}} A_{x_f} - A_{x_f} A_{e_{sf}} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \dots \\ &= \sum A_{x_f} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} - \sum A_{\left(A_{x_f} \begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} \right)} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + \dots \end{aligned} \tag{6-14}$$

and both these terms are zero as a consequency of (6-3). Also the virtual work done by (6-12) in infinitesimal displacements $\delta\phi$, $\delta\theta$, $\delta\psi$ is, using a result from Ref 2, equation (57),

$$[\delta\phi \quad \delta\theta \quad \delta\psi]\sum\left\{A_{x_f} + \begin{bmatrix} [x_f \quad y_f \quad z_f]\dfrac{\delta B_{\phi\theta}}{\delta\phi} \\ [x_f \quad y_f \quad z_f]\dfrac{\delta B_{\phi\theta}}{\delta\theta} \\ [x_f \quad y_f \quad z_f]\dfrac{\delta B_{\phi\theta}}{\delta\psi} \end{bmatrix} + \ldots\right\}\begin{bmatrix} e_s^{(c)} \\ f_s^{(c)} \\ g_s^{(c)} \end{bmatrix}$$

$$= [\delta\phi \quad \delta\theta \quad \delta\psi]\left\{\sum A_{x_f}\begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \left(-\sum A_{x_f}A_{e_{sf}} + \sum A_{e_{sf}}A_{x_f}\right)\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}\right.$$

$$\left. + K_\phi^T \sum A_{x_f}\begin{bmatrix} e_{sf} \\ f_{sf} \\ g_{sf} \end{bmatrix} + \ldots\right\} \qquad (6\text{-}15)$$

where
$$K_\phi = A_\phi - J_\phi \qquad (6\text{-}16)$$

(*cf* equation (2-7)) and again all the exhibited terms are zero (as expected). However the model that gives rise to the non-zero (33) submatrix in (6-10) differs from this in that the expression for the particle coordinates (6-13) is curtailed after the second term. This makes no difference to the first order terms in the expression for the moment, but in the virtual work we are just left with the $\left(-\sum A_{x_f}A_{e_{sf}}\right)$ term. The apparently negligible deformation in the approximately rigid body rotations is therefore sufficient to produce this unexpected structural stiffness.

7 THE EFFECTIVE FORCES

The form of Lagrange's equation appropriate to a non-inertial frame is[1]

$$\frac{\partial V_0}{\partial q_i} + J_i + G_i + \frac{d}{dt}\left(\frac{\partial W}{\partial q_i}\right) - \frac{\partial W}{\partial q_i} = Q_i \qquad (7\text{-}1)$$

011 where, taking the non-inertial frame to be the no-deformation-body-fixed axes,

$$V_0 = \sum \delta m \left\{ \tfrac{1}{2} \begin{bmatrix} p^{(n)} & q^{(n)} & r^{(n)} \end{bmatrix} A^2_{x_n^{(n)}} \begin{bmatrix} p^{(n)} \\ q^{(n)} \\ r^{(n)} \end{bmatrix} \right.$$

$$- \begin{bmatrix} p^{(n)} & q^{(n)} & r^{(n)} \end{bmatrix} A_{x_n^{(n)}} \begin{bmatrix} u^{(n)} \\ v^{(n)} \\ w^{(n)} \end{bmatrix}$$

$$\left. + \begin{bmatrix} x_n^{(n)} & y_n^{(n)} & z_n^{(n)} \end{bmatrix} \begin{bmatrix} \dot{u}^{(n)} \\ \dot{v}^{(n)} \\ \dot{w}^{(n)} \end{bmatrix} \right\} \qquad (7\text{-}2)$$

$$W = \tfrac{1}{2} \sum \delta m \left(\dot{x}_n^{(n)2} + \dot{y}_n^{(n)2} + \dot{z}_n^{(n)2} \right) \qquad (7\text{-}3)$$

$$G_i = 2 \sum \delta m \begin{bmatrix} \dfrac{\partial x_n^{(n)}}{\partial q_i} & \dfrac{\partial y_n^{(n)}}{\partial q_i} & \dfrac{\partial z_n^{(n)}}{\partial q_i} \end{bmatrix} A_{p^{(n)}} \begin{bmatrix} \dot{x}_n^{(n)} \\ \dot{y}_n^{(n)} \\ \dot{z}_n^{(n)} \end{bmatrix} \qquad (7\text{-}4)$$

$$J_i = \sum \delta m \begin{bmatrix} \dfrac{\partial x_n^{(n)}}{\partial q_i} & \dfrac{\partial y_n^{(n)}}{\partial q_i} & \dfrac{\partial z_n^{(n)}}{\partial q_i} \end{bmatrix} A_{\dot{p}^{(n)}} \begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} \qquad (7\text{-}5)$$

Q_i is the generalised applied force obtained by the method of virtual work - in the assumed virtual displacement the frame of reference is regarded as stationary

and q_i is the generalised coordinate of the ith degree of freedom where the freedom is one such that the position and orientation of the frame of reference is independent of it.

The expression on the left hand side of the equation (7-1) is known as the effective force for the degree of freedom q_i . Using the expressions for the velocities of the no-deformation-body-fixed axes (equations (2-2) and (2-3)) and 011
for the location of a particle relative to those axes (equation (2-1)) we then find

that the column vector of the reversed effective forces for the deformational freedoms is

$$\begin{bmatrix} \frac{\partial V_0}{\partial q_1} + J_1 + G_1 + \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_1}\right) - \frac{\partial W}{\partial q_1} \\ \vdots \\ \frac{\partial V_0}{\partial q_n} + J_n + G_n + \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_n}\right) - \frac{\partial W}{\partial q_n} \end{bmatrix} = \left(\sum \delta m R^T R\right)\begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + \left(\sum \delta m R^T\right)\begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} - \left(\sum \delta m R^T A_{x_f}\right)\begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix}$$

$$+ \ldots\ldots \quad . \tag{7-6}$$

Applying the principles of linear and angular momentum respectively in the directions of and about stationary axes which are instantaneously coincident with the no-deformation-body-fixed axes we obtain the relationships

$$\begin{bmatrix} X^{(n)} \\ Y^{(n)} \\ Z^{(n)} \end{bmatrix} = \sum \delta m \left\{\begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p^{(n)}}\begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix}\right\} \tag{7-7}$$

$$\begin{bmatrix} L_n^{(n)} \\ M_n^{(n)} \\ N_n^{(n)} \end{bmatrix} = \sum \delta m A_{x_n^{(n)}} \left\{\begin{bmatrix} \dot{u}_m^{(n)} \\ \dot{v}_m^{(n)} \\ \dot{w}_m^{(n)} \end{bmatrix} + A_{p^{(n)}}\begin{bmatrix} u_m^{(n)} \\ v_m^{(n)} \\ w_m^{(n)} \end{bmatrix}\right\} \tag{7-8}$$

where $X^{(n)}$, $Y^{(n)}$, $Z^{(n)}$ are the resolutes along the no-deformation-body-fixed axes of the total applied force; and $L_n^{(n)}$, $M_n^{(n)}$, $N_n^{(n)}$ are the components about those axes of the total applied torque about their origin. Thus the effective forces for the body freedom equations are given by the right hand sides of the above equations and are (using equations (2-1), (2-3) and (2.8))

(i) for the translational equations

$$\left(\sum \delta m R\right)\begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_n \end{bmatrix} + m\begin{bmatrix} \ddot{x}_1^{(c)} \\ \ddot{y}_1^{(c)} \\ \ddot{z}_1^{(c)} \end{bmatrix} - \left(\sum \delta m A_{x_f}\right)\begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \ldots \tag{7-9}$$

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(ii) for the rotational freedoms

$$\left(\sum \delta m A_{x_f} R\right)\begin{bmatrix}\ddot{q}_1\\ \vdots\\ \ddot{q}_n\end{bmatrix} + \left(\sum \delta m A_{x_f}\right)\begin{bmatrix}\ddot{x}_1^{(c)}\\ \ddot{y}_1^{(c)}\\ \ddot{z}_1^{(c)}\end{bmatrix} - \left(\sum \delta m A_{x_f}^2\right)\begin{bmatrix}\ddot{\phi}\\ \ddot{\theta}\\ \ddot{\psi}\end{bmatrix} + \ldots \quad . \qquad (7\text{-}10)$$

8 THE EQUATIONS OF MOTION

The equations of motion, which can be described as the equation of the applied and effective forces, have already been stated in general terms in section 7 (equations (7-1), (7-7) and (7-8)). These must be satisfied whatever the perturbation; and in particular when the perturbation of the datum motion is zero, and when that perturbation is small.

8.1 The equation of equilibrium

When the perturbations are zero the effective forces are zero and so adding together the various contributions to the generalised and overall forces we find that we must have

$$g\left(\sum \delta m R^T\right)\ell_{\Phi_f} + \sum R^T\begin{bmatrix}e_{sf}\\ f_{sf}\\ g_{sf}\end{bmatrix} + \sum R^T\begin{bmatrix}e_{af}\\ f_{af}\\ g_{af}\end{bmatrix} = 0 \qquad (8\text{-}1)$$

$$mg\ell_{\Phi_f} + \begin{bmatrix}X_{af}\\ Y_{af}\\ Z_{af}\end{bmatrix} = 0 \qquad (8\text{-}2)$$

$$g\left(\sum \delta m A_{x_f}\right)\ell_{\Phi_f} + \begin{bmatrix}L_{af}\\ M_{af}\\ N_{af}\end{bmatrix} = 0 \quad . \qquad (8\text{-}3)$$

8.2 The perturbation motion equations

When we collect together, from sections 4 to 7 the various contributions to the applied and effective, generalised and overall forces, we find that for 01

small perturbations the following equation must be satisfied, in addition to (8-1),

$$\left\{[A_{ij}]D^2 + [G_{ij}] + [E_{ij}] - [Q_{ij}]\right\}\begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi \\ \theta \\ \psi \end{bmatrix} = 0 \qquad (8\text{-}4)$$

where the various constituent matrices are:

(i) <u>Ponderous inertia</u>

$$[A_{ij}] = \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & -\sum \delta m R^T A_{x_f} \\ \sum \delta m R & mI & -\sum \delta m A_{x_f} \\ \sum \delta m A_{x_f} R & \sum \delta m A_{x_f} & -\sum \delta m A_{x_f}^2 \end{bmatrix} \qquad (8\text{-}5)$$

(ii) <u>Gravitational</u>

$$[G_{ij}] = g\begin{bmatrix} 0 & 0 & -\left(\sum \delta m R^T\right) A_{\ell\Phi_f} \\ 0 & 0 & -mA_{\ell\Phi_f} \\ A_{\ell\Phi_f}\left(\sum \delta m R\right) & 0 & -\left(\sum \delta m A_{x_f}\right) A_{\ell\Phi_f} \end{bmatrix} \qquad (8\text{-}6)$$

(iii) Structural

$$[E_{ij}] = \begin{bmatrix} -\sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (8\text{-}7)$$

(iv) Aerodynamics

$$-[Q_{ij}] = -\begin{bmatrix} \sum R^T \begin{bmatrix} e_{a1} & \cdots & e_{an} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \sum R^T \left\{ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & \cdots\cdots & \\ g_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \right\} \\ \begin{bmatrix} X_{a1} & \cdots & X_{an} \\ Y_{a1} & \cdots\cdots & \\ Z_{a1} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} X_{\dot{\phi}} & X_{\dot{\theta}} & X_{\dot{\psi}} \\ Y_{\dot{\phi}} & \cdots\cdots & \\ Z_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \\ \begin{bmatrix} L_{a1} & \cdots & L_{an} \\ M_{a1} & \cdots\cdots & \\ N_{a1} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\phi}} & L_{\dot{\theta}} & L_{\dot{\psi}} \\ M_{\dot{\phi}} & \cdots\cdots & \\ N_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D \end{bmatrix} + \cdots \; . \qquad (8\text{-}8)$$

8.3 Various particularisations of the perturbation motion equations

Of the cases mentioned in section 2.1, probably the one of most interest is that when the modal matrix has the form (2-12), with n increased to $n + 6$ and $\tilde{R}$ put equal to R, and the specific body freedom coordinates are made zero. The perturbation motion equation then is

$$\{[A_{ij}]D^2 + [G_{ij}] + [E_{ij}] - [Q_{ij}]\} \begin{bmatrix} q_i \\ \vdots \\ q_{n+6} \end{bmatrix} = 0 \qquad (8\text{-}9)$$

where the ponderous inertia matrix $[A_{ij}]$ is given by (8-5), and the other matrices are

(ii) <u>Gravitational</u> (*cf* equation (5-7))

$$\left[G_{ij}\right] = 0 \tag{8-10}$$

(iii) <u>Structural</u> (*cf* equation (6-10))

$$\left[E_{ij}\right] = \begin{bmatrix} -\sum R^T \begin{bmatrix} e_{s1} & \cdots & e_{sn} \\ f_{s1} & \cdots\cdots & \\ g_{s1} & \cdots\cdots & \end{bmatrix} & 0 & \sum R^T A_{e_{sf}} \\ 0 & 0 & 0 \\ -\sum A_{e_{sf}} R & 0 & \sum A_{x_f} A_{e_{sf}} \end{bmatrix} \tag{8-11}$$

(iv) <u>Aerodynamic</u> (*cf* section 4)

$$-\left[Q_{ij}\right] = -\begin{bmatrix} \sum R^T \begin{bmatrix} e_{a1} & \cdots & e_{an} \\ f_{a1} & \cdots\cdots & \\ g_{a1} & \cdots\cdots & \end{bmatrix} & \sum R^T \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \sum R^T \left\{ \begin{bmatrix} e_{\dot{x}} & e_{\dot{y}} & e_{\dot{z}} \\ f_{\dot{x}} & \cdots\cdots & \\ g_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} e_{\dot{\phi}} & e_{\dot{\theta}} & e_{\dot{\psi}} \\ f_{\dot{\phi}} & & \\ g_{\dot{\phi}} & & \end{bmatrix} D - A_{e_{af}} \right\} \\ \begin{bmatrix} X_{a1} & \cdots & X_{an} \\ Y_{a1} & \cdots\cdots & \\ Z_{a1} & \cdots\cdots & \end{bmatrix} & \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots\cdots & \\ Z_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} X_{\dot{\phi}} & X_{\dot{\theta}} & X_{\dot{\psi}} \\ Y_{\dot{\phi}} & \cdots\cdots & \\ Z_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D - A_{x_{af}} \\ \begin{bmatrix} L_{a1} & \cdots & L_{an} \\ M_{a1} & \cdots\cdots & \\ N_{a1} & \cdots\cdots & \end{bmatrix} + \sum A_{e_{af}} R & \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} D & \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots\cdots & \\ N_{\dot{x}} & \cdots\cdots & \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\phi}} & L_{\dot{\theta}} & L_{\dot{\psi}} \\ M_{\dot{\phi}} & \cdots\cdots & \\ N_{\dot{\phi}} & \cdots\cdots & \end{bmatrix} D - \sum A_{x_f} A_{e_{af}} \end{bmatrix}$$

$$+ \cdots \quad . \tag{8-12}$$

When one compares this form of the equations of motion with that of the previous section one sees that there are differences in the individual contributions to the (13), (23), (31) and (33) submatrices. However as a consequence of the equilibrium equation (equation (8-2)) the total difference in the second of these submatrices is clearly zero. In the (33) matrix the total difference is

$$g\left(\sum \delta m A_{x_f}\right) A_{\ell_{\Phi_f}} + \sum A_{x_f}\left(A_{e_{af}} + A_{e_{sf}}\right). \tag{8-13}$$

However in the datum state the total force on any particle must be zero* and so

$$\delta m g \ell_{\Phi_f} + \begin{bmatrix} e_{sf} \\ \vdots \end{bmatrix} + \begin{bmatrix} e_{af} \\ \vdots \end{bmatrix} = 0 \tag{8-14}$$

at every particle

and so the above quantity (equation (8-13)) is also zero. Equation (8-14) also makes

$$g\left(\sum \delta m R^T\right) A_{\ell_{\Phi_f}} + \sum R^T\left(A_{e_{af}} + A_{e_{sf}}\right) = 0 \tag{8-15}$$

and so the differences in the (13) and (31) submatrices are also zero.

Finally, in connection with this particularisation, some may think that the appearance of datum values of the applied forces in the various expressions for the coefficients (equations (8-11) and (8-12)) is a denial of one of the advantages claimed (*cf* section 1) for the present approach. However this is just a consequence of writing the typical coefficient matrix

$$\begin{bmatrix} \bar{e}_{n+4} & \bar{e}_{n+5} & \bar{e}_{n+6} \\ \bar{f}_{n+4} & \cdots & \cdots \\ \bar{g}_{n+4} & \cdots & \cdots \end{bmatrix}$$

in terms of the coefficients appearing in the general form (*cf* equations (3-2) and (3-1)).

The above is the form the equations of motion take when the principal frame of reference used is a set of constant-velocity axes. There have however been quite a number of people[11-13] who have advocated using mean-body axes as the principal frame. This means merely taking a set of deformation modes which

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* The equilibrium equations of section (8.1) are only a truly sufficient statement if the semi-rigid model is exact. If (8-14) is satisfied then (8-1) to (8-3) are automatically satisfied but the converse is not necessarily true.

satisfy (2-9). The consequent effect on the general perturbation equations of section 8.2 is that the inertial and gravitational coupling between the deformational and rigid body freedoms is eliminated. With the particular mean-body axes which have their origin at the aircraft CG then (*cf* equation (2-10), the inertial cross-coupling between the translational and rotational body freedoms, and the direct and cross gravitational terms for the rotational body freedoms, are also eliminated.

In section 2.1 the expression of the modal behaviour in terms of the displacements etc of strips, and the way the modes could be chosen to make the no-deformation-body-fixed axes always body-fixed, were also discussed. These particularisations however make no change to the perturbation motion equation of section 8.2 and it is merely a matter of making the appropriate substitution for the modal matrix R .

Appendix

DETERMINATION OF THE PHUGOID FOR THE 'RIGID' AIRCRAFT

At times the ability of the structural dynamicist's approach to predict low frequency, predominantly rigid, perturbations from the aircraft's datum flight state has been questioned. This doubt is clearly a consequence of the fact that the flutter investigator, when considering symmetric perturbations, normally does not include any freedom in fore and aft translation, and so the lowest frequency characteristic solution that he obtains is the short-period. We will show that the much lower frequency phugoid can also be obtained, by considering its determination for the rigid aircraft, using the same approach.

A.1 The structural dynamicist takes constant-velocity axes as his principal frame of reference and so the perturbation motion equation for the 'rigid' aircraft is that given at the beginning of section 8.3 with n = 0. Thus the various matrices are:

(i) Gravitational

$$\left[G_{ij}\right] = 0 \tag{A-1}$$

(ii) Structural

$$\left[E_{ij}\right] = \begin{bmatrix} 0 & 0 \\ 0 & \sum A_{x_f} A_{e_{sf}} \end{bmatrix} \tag{A-2}$$

(iii) Aerodynamic

$$-\left[Q_{ij}\right] = -\begin{bmatrix} \begin{bmatrix} X_{\dot{x}} & X_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots & \cdots \\ Z_{\dot{x}} & \cdots & \cdots \end{bmatrix} D & \begin{bmatrix} X_{\dot{x}} & Y_{\dot{y}} & X_{\dot{z}} \\ Y_{\dot{x}} & \cdots & \cdots \\ Z_{\dot{x}} & \cdots & \cdots \end{bmatrix} A_{u_f} + \begin{bmatrix} X_{\dot{\phi}} & X_{\dot{\theta}} & X_{\dot{\psi}} \\ Y_{\dot{\phi}} & \cdots & \cdots \\ Z_{\dot{\phi}} & \cdots & \cdots \end{bmatrix} D - A_{X_{af}} \\ \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & \cdots & \cdots \\ N_{\dot{x}} & \cdots & \cdots \end{bmatrix} D & \begin{bmatrix} L_{\dot{x}} & L_{\dot{y}} & L_{\dot{z}} \\ M_{\dot{x}} & & \\ N_{\dot{x}} & & \end{bmatrix} A_{u_f} + \begin{bmatrix} L_{\dot{\theta}} & L_{\dot{\theta}} & L_{\dot{\psi}} \\ M_{\dot{\phi}} & \cdots & \cdots \\ N_{\dot{\phi}} & \cdots & \cdots \end{bmatrix} D - \sum A_{x_f} A_{e_{af}} \end{bmatrix}$$

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...... (A-3)

(iv) Inertia

$$[A_{ij}] = \begin{bmatrix} mI & -\sum \delta m A_{x_f} \\ \sum \delta m A_{x_f} & -\sum \delta m A^2_{x_f} \end{bmatrix} . \tag{A-4}$$

As mentioned in section 3, propulsive forces can be treated separately rather than include them in the aerodynamic forces. For our present purpose it is convenient so to do. We will assume that the local propulsive force has constant components, referred to axes fixed in the aircraft, during perfectly rigid perturbations from the datum motion. This means that in the expression for the local propulsive force (*cf* the expression for the typical local force equation (3-2)) the coefficient matrices

$$\begin{bmatrix} e_{px} & e_{py} & e_{pz} \\ f_{px} & \cdots & \cdots \\ g_{px} & \cdots & \cdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e_{p\phi} & e_{p\theta} & e_{p\psi} \\ f_{p\phi} & \cdots & \cdots \\ g_{p\phi} & \cdots & \cdots \end{bmatrix}$$

are both zero. Consequently, by comparison with the derivation of the aerodynamic matrix, or otherwise, we find that we have a propulsive matrix

$$[P_{ij}] = \begin{bmatrix} 0 & A_{X_{pf}} \\ 0 & \sum A_{x_f} A_{e_{pf}} \end{bmatrix} \tag{A-5}$$

and the perturbation motion equation is

$$\left\{[A_{ij}]D^2 + [G_{ij}] + [E_{ij}] + [P_{ij}] - [Q_{ij}]\right\} \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi^* \\ \theta^* \\ \psi^* \end{bmatrix} = 0 \qquad \text{(A-6)}$$

where we have now replaced q_1 to q_6 by symbols which indicate the nature of the perturbations*.

A.2 For the determination of the phugoid we need only consider symmetric perturbations of unbanked ($\Phi_f = 0$) flight, and so putting $y_1^{(c)}$, ϕ^*, ψ^* all equal to zero in (A-6) it becomes

$$\left\{\begin{bmatrix} mD^2 - X_{\dot x}D & -X_{\dot z}D & \left(\sum \delta m z_f\right)D^2 - X_{\dot\theta}D + X_{\dot x}w_f - X_{\dot z}u_f \\ -Z_{\dot x}D & mD^2 - Z_{\dot z}D & -\left(\sum \delta m x_f\right)D^2 - Z_{\dot\theta}D + Z_{\dot x}w_f - Z_{\dot z}u_f \\ \left(\sum \delta m z_f\right)D^2 - M_{\dot x}D & -\left(\sum \delta m x_f\right)D^2 - M_{\dot z}D & \left(\sum \delta m\left(z_f^2 + x_f^2\right)\right)D^2 - M_{\dot\theta}D + M_{\dot x}w_f - M_{\dot z}w_f \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} 0 & 0 & -(Z_{af} + Z_{pf}) \\ 0 & 0 & X_{af} + X_{pf} \\ 0 & 0 & -\sum\left\{(g_{af} + g_{sf} + g_{pf})z_f + (e_{af} + e_{sf} + e_{pf})x_f\right\} \end{bmatrix}\right\} \begin{bmatrix} x_1^{(c)} \\ z_1^{(c)} \\ \theta^* \end{bmatrix} = 0 \qquad \text{(A-7)}$$

Now from the equations of equilibrium ((8-2) and (8-3) with the propulsive forces included) we have, since $\Phi_f = 0$,

$$\left.\begin{aligned} X_{af} + X_{pf} &= mg \sin \Theta_f \\ Z_{af} + Z_{pf} &= -\, mg \cos \Theta_f \end{aligned}\right\} \qquad \text{(A-8)}$$

In addition, as pointed out in section 8.3, the total force on any particle must be zero, and so (*cf* equation (8-14))

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* It is hoped that this will cause no confusion since the specific body freedom
11 coordinates were deleted at a somewhat earlier stage (*cf* section 8.3). The asterisk has been attached to the rotational symbols to indicate that these do not precisely represent rotations.

$$\left.\begin{aligned} e_{af} + e_{sf} + e_{pf} &= \delta mg \sin \Theta_f \\ g_{af} + g_{sf} + g_{pf} &= -\delta mg \cos \Theta_f \end{aligned}\right\} . \tag{A-9}$$

Therefore

$$\left.\begin{aligned} \sum(e_{af} + e_{sf} + e_{pf})x_f &= \left(\sum \delta mx_f\right)g \sin \Theta_f \\ \sum(g_{af} + g_{sf} + g_{pf})z_f &= -\left(\sum \delta mz_f\right)g \cos \Theta_f \end{aligned}\right\} . \tag{A-10}$$

A.3 It is known from experience that during the phugoid the incidence of the aircraft remains approximately constant throughout. We will therefore assume that the perturbations are such that the incidence remains constant. With the perturbation θ^* any finite straight line on the aircraft, in a $y_c^{(c)}\left(= y_n^{(n)}\right) =$ constant plane, remains a straight line but it is rotated through an angle $\tan^{-1}\theta^*$ about the $y_c^{(c)}$ axis as well as undergoing some translation and slight change in length. The incidence is defined by the direction, of the velocity of some reference point on the aircraft, relative to such a line. We will take as the reference point the particle ($x_f = y_f = z_f = 0$). Then since, from equation (2-8), for our present case†, for this particle

$$\left.\begin{aligned} u_m^{(n)} &= u_f + \dot{x}_1^{(c)} \\ w_m^{(n)} &= w_f + \dot{z}_1^{(c)} \end{aligned}\right\} \tag{A-11}$$

we require

$$\frac{u_f}{w_f} = \frac{u_m^{(n)} - w_m^{(n)}\theta^*}{w_m^{(n)} + u_m^{(n)}\theta^*} = \frac{u_f + \dot{x}_1^{(c)} - \left(w_f + \dot{z}_1^{(c)}\right)\theta^*}{w_f + \dot{z}_1^{(c)} + \left(u_f + \dot{x}_1^{(c)}\right)\theta^*} \tag{A-12}$$

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† One has to go through the steps

$$\begin{aligned} x_1^{(c)} = \phi &= 0 \text{ etc} \\ n &= n + 6 \\ R &= \left[\tilde{R} \quad I \quad -A_{xf}\right] \\ n &= 0 \\ q_1 &= x_1^{(c)} \end{aligned}$$

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$$\theta^* = \frac{w_f \dot{x}_1^{(c)} - u_f \dot{z}_1^{(c)}}{u_f^2 + w_f^2 + u_f \dot{x}_1^{(c)} + w_f \dot{z}_1^{(c)}} = \frac{w_f}{V_f^2}\dot{x}_1^{(c)} - \frac{u_f}{V_f^2}\dot{z}_1^{(c)} + \ldots \quad . \qquad \text{(A-13)}$$

For level flight, in which case

$$\left.\begin{aligned} u_f &= V_f \cos \Theta_f \\ w_f &= V_f \sin \Theta_f \end{aligned}\right\} , \qquad \text{(A-14)}$$

this condition becomes

$$\theta^* = \frac{\dot{x}_1^{(c)}}{V_f} \sin \Theta_f - \frac{\dot{z}_1^{(c)}}{V_f} \cos \Theta_f + \ldots \quad . \qquad \text{(A-15)}$$

Putting this condition into the first two equations of (A-7), neglecting the third equation, assuming that in the datum state the aircraft CG coincides with the particle $x_f = y_f = z_f = 0$, and making use of (A-8) and (A-14), gives

$$\begin{bmatrix} \left(m - \frac{X_{\dot\theta}}{V_f}\sin\Theta_f\right)D^2 - \left\{X_{\dot x}\cos^2\Theta_f + \left(X_{\dot z} - \frac{mg}{V_f}\right)\sin\Theta_f\cos\Theta_f\right\}D & \left(\frac{X_{\dot\theta}}{V_f}\cos\Theta_f\right)D^2 - \left\{X_{\dot z}\sin^2\Theta_f + X_{\dot x}\sin\Theta_f\cos\Theta_f + \frac{mg}{V_f}\cos^2\Theta_f\right\}D \\ \left(-\frac{Z_{\dot\theta}}{V_f}\sin\Theta_f\right)D^2 - \left\{Z_{\dot x}\cos^2\Theta_f + Z_{\dot z}\sin\Theta_f\cos\Theta_f - \frac{mg}{V_f}\sin^2\Theta_f\right\}D & \left(m + \frac{Z_{\dot\theta}}{V_f}\cos\Theta_f\right)D^2 - \left\{Z_{\dot z}\sin^2\Theta_f + \left(Z_{\dot x} + \frac{mg}{V_f}\right)\sin\Theta_f\cos\Theta_f\right\}D \end{bmatrix} \begin{bmatrix} x_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \approx 0$$

...... (A-16)

For our present purpose it will be good enough to assume that the two $\dot{\theta}$ coefficients, $X_{\dot{\theta}}$ and $Z_{\dot{\theta}}$, are zero and that the other aerodynamic coefficients are constant. We then see that the two non-trivial solutions of (A-16) have a circular frequency ω given by

$$\omega = \frac{1}{m}\sqrt{\begin{aligned}&-\left(X_{\dot{z}} \sin^2\Theta_f + X_{\dot{x}} \sin\Theta_f \cos\Theta_f + \frac{mg}{V_f}\cos^2\Theta_f\right)\left(Z_{\dot{x}}\cos^2\Theta_f + Z_{\dot{z}}\sin\Theta_f\cos\Theta_f - \frac{mg}{V_f}\sin^2\Theta_f\right)\\ &-\frac{1}{4}\left\{X_{\dot{x}}\cos^2\Theta_f - Z_{\dot{z}}\sin^2\Theta_f + \left(X_{\dot{z}} - Z_{\dot{x}} - \frac{2mg}{V_f}\right)\sin\Theta_f\cos\Theta_f\right\}^2\end{aligned}}$$

$$= \frac{1}{m}\sqrt{\begin{aligned}&-\left(X_{\dot{z}}\sin^2\Theta_f + X_{\dot{x}}\sin\Theta_f\cos\Theta_f\right)\left(Z_{\dot{x}}\cos^2\Theta_f + Z_{\dot{z}}\sin\Theta_f\cos\Theta_f\right)\\ &-\frac{1}{4}\left\{X_{\dot{x}}\cos^2\Theta_f - Z_{\dot{z}}\sin^2\Theta_f + (X_{\dot{z}} - Z_{\dot{x}})\sin\Theta_f\cos\Theta_f\right\}^2\\ &+\frac{mg}{V_f}\left\{X_{\dot{z}}\sin^2\Theta_f + (X_{\dot{x}} - Z_{\dot{z}})\sin\Theta_f\cos\Theta_f - Z_{\dot{x}}\cos^2\Theta_f\right\}\end{aligned}} \qquad \text{(A-17)}$$

The temporal behaviour of the solution is $e^{\mu t}\cos(\omega t + \text{constant})$

where $$\mu = \frac{1}{2m}\left\{X_{\dot{x}}\cos^2\Theta_f + (X_{\dot{z}} + Z_{\dot{x}})\sin\Theta_f\cos\Theta_f + Z_{\dot{z}}\sin^2\Theta_f\right\}. \qquad \text{(A-18)}$$

To quantify the above solution let us take a simple aerodynamic model which accords with the assumptions that we have already made. Thus we assume that the aerodynamic force consists of (i) a force acting normal to the direction of motion of the reference particle ($x_f = y_f = z_f = 0$) which is a function of the velocity of that particle and the incidence, and (ii) a force acting in the direction of motion of the reference particle which is a function of the incidence times the $(\text{velocity})^2$ of that particle. In particular we write the former as

$$N = V_0^2\left\{N_f + \frac{\partial N_f}{\partial V_f}(V_0 - V_f) + \frac{\partial N_f}{\partial \Theta_f}(\alpha_0 - \alpha_f)\right\} \qquad \text{(A-19)}$$

and the latter as $-D$

where $$D = V_0^2\left\{D_f + \frac{dD_f}{d\Theta_f}(\alpha_0 - \alpha_f)\right\} \qquad \text{(A-20)}$$

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and (*cf* equations (A-11) and (A-12))

$$V_0 = \sqrt{\left(u_f + \dot{x}_1^{(c)}\right)^2 + \left(w_f + \dot{z}_1^{(c)}\right)^2} \tag{A-21}$$

$$\alpha_f = \cot^{-1}\left(\frac{u_f}{w_f}\right) \tag{A-22}$$

$$\alpha_0 = \cot^{-1}\left\{\frac{u_f + \dot{x}_1^{(c)} - \left(w_f + \dot{z}_1^{(c)}\right)\theta^*}{w_f + \dot{z}_1^{(c)} + \left(u_f + \dot{x}_1^{(c)}\right)\theta^*}\right\} . \tag{A-23}$$

The components of the overall aerodynamic force in the directions of the constant-velocity axes then are

$$\left.\begin{aligned} X_a^{(n)} &= -\frac{u_f + \dot{x}_1^{(c)}}{V_0} D - \frac{w_f + \dot{z}_1^{(c)}}{V_0} N \\ Z_a^{(n)} &= -\frac{w_f + \dot{z}_1^{(c)}}{V_0} D + \frac{u_f + \dot{x}_1^{(c)}}{V_0} N \end{aligned}\right\} \tag{A-24}$$

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$$\left.\begin{aligned} X_a^{(n)} &= V_0(- u_f D_f - w_f N_f) \\ &\quad + V_f\left(- \dot{x}_1^{(c)} D_f - \dot{z}_1^{(c)} N_f - \left\{u_f \frac{dD_f}{d\Theta_f} + w_f \frac{\partial N_f}{\partial \Theta_f}\right\}(\alpha_0 - \alpha_f)\right. \\ &\qquad \left. - w_f \frac{\partial N_f}{\partial V_f}(V_0 - V_f)\right) + \ldots \\ Z_a^{(n)} &= V_0(- w_f D_f + u_f N_f) \\ &\quad + V_f\left(- \dot{z}_1^{(c)} D_f + \dot{x}_1^{(c)} N_f + \left\{- w_f \frac{dD_f}{d\Theta_f} + u_f \frac{\partial N_f}{\partial \Theta_f}\right\}(\alpha_0 - \alpha_f)\right. \\ &\qquad \left. + u_f \frac{\partial N_f}{\partial V_f}(V_0 - V_f)\right) + \ldots \quad . \end{aligned}\right\} \tag{A-25}$$

Now

$$V_0 = V_f\left(1 + \frac{u_f\dot{x}_1^{(c)}}{V_f^2} + \frac{w_f\dot{z}_1^{(c)}}{V_f^2} + \ldots\right) \tag{A-26}$$

$$V_0 - V_f = \frac{u_f\dot{x}_1^{(c)}}{V_f} + \frac{w_f\dot{z}_1^{(c)}}{V_f} + \ldots \tag{A-27}$$

$$\alpha_0 - \alpha_f = \theta^* - \frac{w_f}{V_f^2}\,\dot{x}_1^{(c)} + \frac{u_f}{V_f^2}\,\dot{z}_1^{(c)} + \ldots \tag{A-28}$$

and so, since (*cf* equations (3-6) and (4-4))

$$\left.\begin{aligned} X_a^{(n)} &= X_{af} + X_{\dot{x}}\dot{x}_1^{(c)} + X_{\dot{z}}\dot{z}_1^{(c)} + (X_{\dot{z}}u_f - X_{\dot{x}}w_f + Z_{af})\theta^* \\ Z_a^{(n)} &= Z_{af} + Z_{\dot{x}}\dot{x}_1^{(c)} + Z_{\dot{z}}\dot{z}_1^{(c)} + (Z_{\dot{z}}u_f - Z_{\dot{x}}w_f - X_{af})\theta^* \end{aligned}\right\} \tag{A-29}$$

we have

$$X_{af} = -\,V_f(u_fD_f + w_fN_f) \tag{A-30}$$

$$Z_{af} = V_f(-\,w_fD_f + u_fN_f) \tag{A-31}$$

$$X_{\dot{x}} = \frac{1}{V_f}\left\{-\left(u_f^2 + V_f^2\right)D_f - u_fw_f\left(N_f - \frac{dD_f}{d\Theta_f} + V_f\,\frac{\partial N_f}{\partial V_f}\right) + w_f^2\,\frac{\partial N_f}{\partial \Theta_f}\right\} \tag{A-32}$$

$$X_{\dot{z}} = \frac{1}{V_f}\left\{-\,u_fw_f\left(D_f + \frac{\partial N_f}{\partial \Theta_f}\right) - \left(w_f^2 + V_f^2\right)N_f - u_f^2\,\frac{dD_f}{d\Theta_f} - w_f^2V_f\,\frac{\partial N_f}{\partial V_f}\right\} \tag{A-33}$$

$$Z_{\dot{x}} = \frac{1}{V_f}\left\{-\,u_fw_f\left(D_f + \frac{\partial N_f}{\partial \Theta_f}\right) + \left(u_f^2 + V_f^2\right)N_f + w_f^2\,\frac{dD_2}{d\Theta_f} + u_f^2V_f\,\frac{\partial N_f}{\partial V_f}\right\} \tag{A-34}$$

$$Z_{\dot{z}} = \frac{1}{V_f}\left\{-\left(w_f^2 + V_f^2\right)D_f + u_fw_f\left(N_f - \frac{dD_f}{d\Theta_f} + V_f\,\frac{\partial N_f}{\partial V_f}\right) + u_f^2\,\frac{\partial N_f}{\partial \Theta_f}\right\}. \tag{A-35}$$

With these values it can be confirmed that the coefficients of θ^* in (A-25) and (A-29) also agree, as, of course, they should.

A.5 We can proceed to evaluate the frequency and decay rate of the phugoid from equations (A-17) and (A-18) when u_f, w_f are given by (A-14), we find that

$$X_{\dot z} \sin^2\Theta_f + X_{\dot x} \sin\Theta_f \cos\Theta_f = -2V_f\left(D_f \sin\Theta_f \cos\Theta_f + N_f \sin^2\Theta_f\right) - V_f^2 \frac{\partial N_f}{\partial V_f} \sin^2\Theta_f \qquad \text{(A-36)}$$

$$Z_{\dot x} \cos^2\Theta_f + Z_{\dot z} \sin\Theta_f \cos\Theta_f = -2\,V_f\left(D_f \sin\Theta_f \cos\Theta_f - N_f \cos^2\Theta_f\right) + V_f^2 \frac{\partial N_f}{\partial V_f} \cos^2\Theta_f \qquad \text{(A-37)}$$

$$X_{\dot x} \cos^2\Theta_f - Z_{\dot z} \sin^2\Theta_f + (X_{\dot z} - Z_{\dot x}) \sin\Theta_f \cos\Theta_f$$
$$= -2V_f\left\{D_f\left(\cos^2\Theta_f - \sin^2\Theta_f\right) + N_f(2\sin\Theta_f \cos\Theta_f)\right\} - V_f^2 \frac{\partial N_f}{\partial V_f}(2\sin\Theta_f \cos\Theta_f) \qquad \text{(A-38)}$$

$$X_{\dot z} \sin^2\Theta_f + (X_{\dot x} - Z_{\dot z}) \sin\Theta_f \cos\Theta_f - Z_{\dot x} \cos^2\Theta_f = -2V_f N_f - V_f^2 \frac{\partial N_f}{\partial V_f} \qquad \text{(A-39)}$$

and

$$X_{\dot x} \cos^2\Theta_f + (X_{\dot z} + Z_{\dot x}) \sin\Theta_f \cos\Theta_f + Z_{\dot z} \sin^2\Theta_f = -2V_f D_f \,. \qquad \text{(A-40)}$$

Consequently, from (A-17) and (A-18)

$$\omega = \frac{V_f}{m}\sqrt{-D_f^2 - \frac{mg}{V_f}\left\{2N_f + V_f \frac{\partial N_f}{\partial V_f}\right\}} \qquad \text{(A-41)}$$

and

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$$\mu = -\frac{V_f D_f}{m} \,. \qquad \text{(A-42)}$$

Bearing in the mind that we have (from (A-8), (A-14), (A-30) and (A-31))

$$N_f = \frac{X_{pf} \sin \Theta_f - Z_{pf} \cos \Theta_f - mg}{V_f^2} \tag{A-43}$$

it is more revealing to rewrite the expression for the frequency in the form

$$\omega = \frac{g}{V_f} \sqrt{2 - \frac{V_f^3}{mg}\frac{\partial N_f}{\partial V_f} - \frac{2}{mg}\left(N_f + \frac{mg}{V_f^2}\right) - \left(\frac{D_f V_f^2}{mg}\right)^2} \quad . \tag{A-44}$$

This reduces to Lanchester's classical solution ($\omega = (g/V_f)\sqrt{2}$; $\mu = 0$) when the propulsive force is zero and $\partial N_f / \partial V_f$ is zero; for, similar to (A-43).

$$D_f = \frac{X_{pf} \cos \Theta_f + Z_{pf} \sin \Theta_f}{V_f^2} \tag{A-45}$$

and so this, and μ , are zero.

If only $\partial N_f / \partial V_f$ and $(N_f + (mg/V_f^2))$ are zero the above solution ((A-44) and (A-42)) is the 'better approximation' obtained by Etkin* (Ref 12, p 331). Without either of these assumptions we have an even better approximation to the true solution; and it is clear that as accurate a determination of the phugoid can be obtained by the present approach as has been obtained by any other. We made a number of simplifying assumptions - constant incidence, no genuine deformational freedoms, simple aerodynamic model and so on - as our purpose was merely to demonstrate the adequacy of the suggested approach rather than make an accurate calculation for some particular case. When one is concerned with this latter aspect the above assumptions can be rejected as desired and indeed in numerical work there is probably little to be gained by making most of them.

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* He writes the damping ratio and frequency in terms of the derivative C_{T_V} of the thrust coefficient with respect to the (airspeed/unperturbed airspeed), but this is a little deceptive for he has already assumed (pages earlier) that the thrust is constant and so C_{T_V} is simply related to the thrust coefficient C_T :

$$C_{T_V} = -2C_T$$

GLOSSARY OF NOMENCLATURE

(i) Frames of reference (all right-handed orthogonal cartesian)

Body-fixed axes	axes whose origin and orientation are fixed in a small material portion of the body
Body-path axes	axes whose origin is at a material point of the body and such that the velocity of that point is in the direction of the x axis.
Constant-velocity axes	axes having constant linear and angular velocity relative to an inertial frame. (The ones used have zero angular velocity.)
Mean-body axes	axes orientated so that the kinetic energy relative to the axes is a minimum.
No-deformation-body-fixed axes	an arbitrary non-inertial frame. Any perturbation can be visualised as a rigid body perturbation which moves the body-fixed axes from coincidence with the constant-velocity axes to coincidence with these axes, followed by a deformation.
Normal earth-fixed axes	axes fixed relative to the earth with the z-axis vertically downwards.
Principal axes of inertia	axes with origin at the centre of gravity and such that the three products of inertia about the axes are zero.
Strip-attached axes	axes with origin at a material point of a strip (the strip reference point) and such that, in our strip model, the deformation of the main part of the strip relative to these axes is of second order of smallness.

(ii) Miscellaneous

Aircraft reference point	the material point on the aircraft which is the origin of the body-fixed axes.
Axes transformation matrix	the matrix which by pre-multiplying a column vector changes the axes directions in respect to which the vector is resolved into components.
Strip	a narrow slice of the aircraft lying more or less for and aft and consisting of one or two almost rigid portions.
Strip reference point	a chosen material point on the main part of a strip taken as the origin of the strip-attached axes.

LIST OF SYMBOLS

A_ϕ etc	skew-symmetric matrices involving ϕ, θ, ψ etc (see equation (2-6))
A_{ij}	ponderous inertia coefficient
B_i	aerodynamic hinge moment on ith strip
$B_{i\dot{x}}$, $B_{i\dot{\theta}}$ etc	aerodynamic hinge moment coefficients for ith strip

$$B_{\phi\theta} = \begin{bmatrix} -\frac{1}{2}(\psi^2 + \theta^2) & 0 & 0 \\ \phi\theta & -\frac{1}{2}(\phi^2 + \psi^2) & 0 \\ \phi\psi & \theta\psi & -\frac{1}{2}(\phi^2 + \theta^2) \end{bmatrix}$$

D	differential operator d/dt
$-E_i$	generalised structural force
E_{ij}	structural stiffness coefficient
F_i	rotational (approximate) modal matrix for ith strip
$-G_i$	generalised gravitational force
G_i	gyrostatic force (*cf* equation (7-4))
G_{ij}	gravitational stiffness coefficient
I	unit matrix (3 × 3)
J_i	a certain coupling force between the rotational body freedoms and the deformational freedoms in Lagrange's equations referred to a non-inertial frame (see equation (7-5))
J_ϕ	a certain matrix formed from the elements of $\{\phi\ \theta\ \psi\}$ - see equation (2-7)
K_i	translational model matrix for the ith strip
$K_\phi = A_\phi - J_\phi$	
L	rolling moment
L_i	rolling moment on ith strip rolling moment coefficient (aerodynamic)
$L_{\dot{x}}$, $L_{\dot{\phi}}$ etc	aerodynamic rolling moment coefficients
M	pitching moment
M_i	pitching moment on ith strip pitching moment coefficient (aerodynamic)
$M_{\dot{x}}$, $M_{\dot{\phi}}$ etc	aerodynamic pitching moment coefficients
N	yawing moment
N_i	yawing moment on ith strip 01 yawing moment coefficient (aerodynamic)

LIST OF SYMBOLS (continued)

Q_j	generalised force
Q_ϕ	matrix relating angular velocities and orientation = $\begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}$
Q_{ij}	aerodynamic coefficient
R	modal matrix (see equation (2-1))
S, S_{Φ_f}	axes transformation matrix. Absence of a subscript means the arguments are ϕ, θ, ψ, $S = I - A_\phi + B_{\phi\theta} \; \ldots$
V_0	centrifugal potential function
W	kinetic energy relative to the frame of reference
X, Y, Z	overall force resolutes
X_i, Y_i, Z_i	$\begin{cases} \text{overall force resolutes on ith strip} \\ \text{overall force resolute coefficients (aerodynamic)} \end{cases}$
$\left.\begin{matrix} X_{\dot{x}}, X_{\dot{\phi}} \text{ etc} \\ Y_{\dot{x}}, Y_{\dot{\phi}} \text{ etc} \\ Z_{\dot{x}}, Z_{\dot{\phi}} \text{ etc} \end{matrix}\right\}$	aerodynamic force resolute coefficients - they may be differential operators
e, f, g	components of local force vector
$\left.\begin{matrix} e_i, e_x, e_\phi \\ f_i, f_x, f_\phi \\ g_i, g_x, g_\phi \end{matrix}\right\}$ etc	local force vector coefficients - they may be differential operators
$\left.\begin{matrix} e_{\dot{x}}, e_{\dot{\phi}} \\ f_{\dot{x}}, f_{\dot{\phi}} \\ g_{\dot{x}}, g_{\dot{\phi}} \end{matrix}\right\}$ etc	local aerodynamic force vector coefficients - they may be differential operators
f_i	flap modal vector at ith strip
g	acceleration due to gravity
ℓ_{Φ_f}	third column of S_{Φ_f}
m	mass of the aircraft
δm	mass of a particle
n	number of deformational degrees of freedom
p, q, r	angular velocity resolutes
q_i	generalised coordinate
t	time
u, v, w	linear velocity resolutes
x, y, z	particle position resolutes

LIST OF SYMBOLS (continued)

x_1, y_1, z_1	position resolutes for origin of no-deformation-body-fixed axes relative to the origin of the constant-velocity axes
Θ_f	angle of inclination in datum motion } orientation angles of constant-velocity axes relative to normal earth-fixed axes
Φ_f	angle of bank in datum motion } orientation angles of constant-velocity axes relative to normal earth-fixed axes
Ψ_f	nose-azimuth angle in datum motion } orientation angles of constant-velocity axes relative to normal earth-fixed axes
δ	angle of flap rotation (approximate)
ϕ, θ, ψ	orientation angles of no-deformation-body-fixed axes relative to constant-velocity axes
ϕ_{if}, θ_{if}, ψ_{if}	orientation angles of the strip-attached axes, relative to the no-deformation-body-fixed axes, in the datum state

Dressings

(i) Subscripts

A quantity is relative to or about the origin of a particular set of axes where

c denotes the constant-velocity axes

h denotes axes with the same orientation as the strip-attached axes but with origin at the strip hinge

n denotes the no-deformation-body-fixed axes

s denotes the strip-attached axes

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a indicates aerodynamic

f indicates the value during the datum motion

g indicates gravitational

i indicates the ith strip

s indicates structural

(ii) Superscripts

Bracketed superscripts denote the axes of resolution with the same significance as in (i) above, also

" indicates a summation over the flap part of a strip

T denotes the transpose of a matrix

LIST OF SYMBOLS (concluded)

(iii) Suprascripts

$\cdot$	(dot)	denotes derivative with respect to time
$\sim$	(tilde)	is of ephemeral significance
$-$	(bar)	denotes typical

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